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# Changes in Mathematics and the Philosophy of Mathematics

by

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Abstract: Several developments are coming in physics and biology that require the creation of new mathematics. This involves a change in the approach of mathematics to these disciplines, from the top-down approach to the bottom-up approach. Mainstream philosophy of mathematics is unable to account for this change. This article argues that, to account for it, a shift is necessary in the philosophy of mathematics, from mainstream philosophy of mathematics to heuristic philosophy of mathematics. This involves moving, from the view that there is a sharp distinction between pure mathematics and applied mathematics, to the view that such distinction does not hold for mathematics created by the bottom-up approach. More importantly, it involves moving, from the view that mathematics is theorem proving by the axiomatic method, to the view that mathematics is problem solving by the analytic method.

Keywords: Top-Down Approach, Bottom-Up Approach, Analytic Method, Mainstream Philosophy of Mathematics, Heuristic Philosophy of Mathematics

Abstract: In fisica e biologia stanno avendo luogo vari sviluppi che richiedono la creazione di nuova matematica. Questo comporta un cambiamento nell'approccio della matematica a queste discipline, passando dall'approccio top-down a quello bottom-up. La filosofia della matematica corrente non è in grado di rendere conto di questo cambiamento. Questo articolo sostiene che, per renderne conto è necessario un cambiamento nella filosofia della matematica, passando dalla filosofia della matematica corrente alla filosofia della matematica euristica. Questo comporta passare dalla concezione secondo cui esiste una netta distinzione tra matematica pura e matematica applicata a quella secondo cui tale distinzione non vale per la matematica creata dall'approccio bottom-up. Cosa ancora più importante, comporta passare dalla concezione secondo cui la matematica è dimostrazione di teoremi con il metodo assiomatico a quella secondo cui la matematica è soluzione di problemi con il metodo analitico.

Keywords: approccio top-down, approccio bottom-up, metodo analitico, filosofia della matematica corrente, filosofia della matematica euristica

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## 1. *Introduction*

Several developments are coming in physics and biology that require the creation of new mathematics. This involves a change in the approach of mathematics to these disciplines, from the top-down approach to the bottom-up approach.

Mainstream philosophy of mathematics is unable to deal with this change because, as Hersh says, «the philosophy of mathematics as practiced in many articles and books is a thing unto itself, hardly connected» to «living mathematics»[1]. Some philosophers of mathematics are even «unfamiliar with anything beyond arithmetic and elementary geometry»[2].

Even some mathematicians seem to be unaware that the change in the approach of mathematics to physics and biology, from a top-down to a bottom-up approach, involves a change in the understanding of mathematics. This is because, as Lovász says, we «mathematicians are conservative people», we «don't push for changes», we «pretend that mathematical research is as it used to be»[3].

In particular, some mathematicians have ideas about mathematics that are a holdover from early twentieth-century ideas advanced by Hilbert and Bourbaki.

Thus, Naylor and Sell say: «The axiomatic method» not only «is the method of mathematics», but in fact «it is mathematics»[4].

Mac Lane says: «The traditional nature of a proof remains. It is a deduction from suitable axioms», the nature of «proof is eternal»[5].

Nathanson says: «We mathematicians have a naive belief in truth. We prove theorems. Theorems are deductions from axioms. Each line in a proof is a simple consequence of the previous lines of the proof, or of previously proved theorems. Our conclusions are true, unconditionally and eternally»[6].

[1] R. Hersh, *Experiencing Mathematics. What Do We Do, When We Do Mathematics?*, American Mathematical Society, Providence 2014, p. 68.
[2] R. Hersh, *What Is Mathematics, Really?*, Oxford University Press, Oxford 1997, p. 41.
[3] L. Lovász, *One Mathematics*, «DMV-Mitteilungen» 2 (1998), pp. 33-39, p. 33.
[4] A. Naylor-G. Sell, *Linear Operator Theory in Engineering and Science*, Springer, Cham 1982, p. 6.
[5] S. Mac Lane, *Proof Is Eternal*, «Proceedings of the American Philosophical Society» 128 (1984), pp. 44-47, p. 44.
[6] M. Nathanson, *Desperately Seeking Mathematical Truth*, «Notices of the American Mathematical Society» 55 (2008), p. 773.

This article argues that, to deal with the change in the approach of mathematics to physics and biology, a shift is necessary in the philosophy of mathematics, from mainstream philosophy of mathematics to heuristic philosophy of mathematics. This involves moving, from the view that there is a sharp distinction between pure mathematics and applied mathematics, to the view that such distinction does not hold for mathematics created by the bottom-up approach. More importantly, it involves moving, from the view that mathematics is theorem proving by the axiomatic method, to the view that mathematics is problem solving by the analytic method.

## 2. *Pure Mathematics and Applied Mathematics*

In mathematics, several distinctions are made. One of them is the distinction between pure mathematics and applied mathematics.

Pure mathematics is mathematics developed for its own sake, without any relation to the real world. Applied mathematics is mathematics developed with relation to the real world.

Some mathematicians also believe that the distinction between pure and applied mathematics is the most important one in mathematics.

Thus, Lovász says that in mathematics there are many distinctions, but «the most prominent of these runs between pure and applied mathematics»[7].

This belief is based on the assumption that pure mathematics is real mathematics, it fully expresses the nature of mathematics, applied mathematics adds nothing to the nature of mathematics.

Thus, Trudeau says: «Pure mathematics is real mathematics. To understand what mathematics is, you need to understand what pure mathematics is»[8]. Applied mathematics is not an independent discipline, because pure mathematics is «the "mathematics" part of applied mathematics»[9]. In fact, «you can see the pure mathematics beneath the applications if you look hard enough»[10].

An extreme version of the view that pure mathematics is real mathematics and applied mathematics adds nothing to the nature of

[7] L. Lovász, *One Mathematics*, cit., p. 33.
[8] R. Trudeau, *Introduction to Graph Theory*, Dover, New York 1993, p. 1.
[9] *Ibidem*.
[10] Ivi, p. 2.

mathematics is that of Hardy, who says: «The "real" mathematics of the "real" mathematicians» is «almost wholly 'useless'»[11]. The mathematics that is useful «is what I will call the 'trivial' mathematics»[12]. In fact, «it is not possible to justify the life of any genuine professional mathematician on the ground of the 'utility' of his work»[13].

However, the distinction between pure mathematics and applied mathematics is problematic. Indeed, it has negative effects on mathematics.

As Kline says, with this distinction pure mathematics is «turned inward; it feeds on itself», it becomes «an end in itself with no thought of what objective it might serve»[14]. Pure mathematicians are «like the mathematicians Gulliver met on his voyage to Laputa», who «live on an island suspended in the air above the earth» and «are doomed ultimately to expire in a vacuum»[15]. But science will not «be deprived of mathematics» because of this, «the Newtons, Laplaces, and Hamiltons of the future will create the mathematics they will need just as they have, in the past», and «these men, though honored as mathematicians, were physicists»[16].

## 3. *Top-Down Approach and Bottom-Up Approach*

That the distinction between pure mathematics and applied mathematics is problematic does not mean, however, that in mathematics there are no significant distinctions.

A significant distinction is that between the top-down approach and the bottom-up approach to non-mathematical fields, in particular physics and biology.

The top-down approach applies already existent mathematics to a non-mathematical field.

The bottom-up approach starts from problems of a non-mathematical field and creates new mathematics from them.

[11] G. Hardy, *A Mathematician's Apology*, Cambridge University Press, Cambridge 1992, p. 119.
[12] Ivi, p. 139.
[13] Ivi, pp. 119-120.
[14] M. Kline, *Mathematics: The Loss of Certainty*, Oxford University Press, Oxford 1980, p. 304.
[15] Ivi, p. 305.
[16] Ivi, pp. 304-305.

An example of the top-down approach is Einstein's general theory of relativity, which applied the already existent Riemann's geometry to the physics of space and time.

As Hilbert observes, Einstein's general theory of relativity «would not have been possible without the profound and difficult mathematical investigations of Riemann, which existed long before»[17].

An example of the bottom-up approach is Newton's creation of calculus.

Indeed, Newton says: «Mathematical quantities I here consider» as «described by a continuous motion. Lines are described and by describing generated» through «the continuous motion of points; surface-areas are through the motion of lines, solids through the motion of surface-areas, angles through the rotation of sides», and «the like in other cases. These geneses take place in the reality of physical nature»[18]. On this basis, Newton creates «a method of determining quantities out of the speeds of motion or increment by which they are generated», and calls «these speeds of motion or increment "fluxions" and the quantities so born "fluents"»[19].

According to Einstein, Newton's creation of calculus is «perhaps the greatest advance in thought that a single individual was ever privileged to make»[20].

In the top-down approach, the already existent mathematics which is applied to a non-mathematical field can be mathematics originally developed with no application in mind.

Thus, Hilbert says: «I developed my theory of infinitely many variables from purely mathematical interests», and «even called it "spectral analysis" without any presentiment that it would later find an application to the actual spectrum of physics»[21].

In the last century the top-down approach has been dominant. Nevertheless, there have been some uses of the bottom-up approach.

An example is Turing's creation of the theory of computation.

[17] D. Hilbert, *Logic and the Knowledge of Nature*, in W. Ewald (ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Oxford University Press, Oxford 1996, vol. II, pp. 1157-1165, p. 1160.

[18] I. Newton, *The 1704 De Quadratura Curvarum. Final Text Additions*, in Id., *The Mathematical Papers*, Cambridge University Press, Cambridge 1981, vol. VIII, pp. 122-159, p. 123.

[19] *Ibidem*.

[20] A. Einstein, *Essays in Science*, Dover, Mineola 2009, p. 42.

[21] C. Reid, *Hilbert-Courant*, Springer, Cham 1986, p. 83.

Turing starts from an analysis of the operations of a human being who makes computations, then he says: «It is my contention that these operations include all those which are used in the computation of a number», on this basis I «proceed with the development of the theory»[22].

## 4. *Developments in Physics and Biology*

While in the last century the top-down approach was dominant, the situation is likely to change in this century.

For, several developments are coming in physics that require new mathematics.

Thus, Atiyah, Dijkgraaf and Hitchin say: «Over the past 30 years» physicists «have stumbled across a whole range of mathematical "discoveries"» which «are derived by physical intuition and heuristic arguments» and «are beyond the reach, as yet, of mathematical rigour, but which have withstood the tests of time and alternative methods»[23]. What «we are now witnessing» is «one of the most refreshing events in the mathematics of the 20th century» and «it might well come to dominate the mathematics of the 21st century»[24].

Several developments are also coming in biology that require new mathematics.

Thus, Burini, Chouhad, and Bellomo say: «The development of mathematical and physical tools to describe the dynamics of living organisms is one of the challenging scientific objectives of this century»[25]. It «is a challenging quest» which «will lead to a new mathematical theory combined with new interpretations of the physics of living systems»[26].

These developments in physics and biology are characterized by a high degree of complexity. Therefore, the new mathematics is unlikely to be created by the top-down approach, it is more likely to

[22] A. Turing, *Collected Works: Mathematical Logic*, North-Holland, Amsterdam 2001, p. 20.
[23] M. Atiyah-R. Dijkgraaf-N. Hitchin, *Geometry and Physics*, «Philosophical Transactions of the Royal Society A» 368 (2010), pp. 913-926, pp. 914-915.
[24] M. Atiyah, *Response to "Theoretical Mathematics. Toward a Cultural Synthesis of Mathematics and Theoretical Physics" by A. Jaffe-F. Quinn*, «Bulletin of the American Mathematical Society» 30 (1994), pp. 178-179, p. 179.
[25] D. Burini-N. Chouhad-N. Bellomo, *Waiting for a Mathematical Theory of Living Systems from a Critical Review to Research Perspectives*, «Symmetry» 15/2 (2023), 351, p. 1.
[26] *Ibidem*.

be created by the bottom-up approach, because only mathematics created by the bottom-up approach can deal with such a degree of complexity. For it starts from problems of physics or biology and creates new mathematics from them.

This involves a change in the approach of mathematics to physics and biology, from the top-down approach to the bottom-up approach.

Of course, there will always be mathematicians who pursue mathematics for its own sake, without any relation to the real world. But social pressure is likely to increasingly demand that mathematicians produce useful mathematics.

Moreover, the real world is a powerful stimulus for the creation of new mathematics, and without this stimulus mathematics can wither away.

## 5. *First Implication of the Change*

The change from the top-down approach to the bottom-up approach has two implications.

The first implication concerns the distinction between pure mathematics and applied mathematics.

With the top-down approach there is a sharp distinction between pure mathematics and applied mathematics. For pure mathematics is created independently of non-mathematical fields.

On the contrary, with the bottom-up approach the distinction between pure mathematics and applied mathematics fades. For the bottom-up approach starts from problems of a non-mathematical field and creates new mathematics from them. So, mathematics thus created is not mathematics developed for its own sake, without any relation to the real world.

Therefore, the change from the top-down approach to the bottom-up approach involves moving, from the view that there is a sharp distinction between pure mathematics and applied mathematics, to the view that such distinction does not hold for mathematics created by the bottom-up approach.

## 6. *Second Implication of the Change*

The second implication of the change from the top-down approach to the bottom-up approach concerns the method of mathematics. The change involves a change in the method of mathematics from the axiomatic method to the analytic method.

According to the axiomatic method, to prove a statement, one starts from a few statements whose truth is taken for granted (axioms) and derives the statement from them by deductive rules.

An example of use of the axiomatic method is Bourbaki's series of volumes *Elements of Mathematics*.

Indeed, Bourbaki says that the «method of exposition we have chosen is axiomatic»[27]. This choice «has been dictated by the main purpose of the treatise, which is to provide a solid foundation for the whole body of modern mathematics»[28].

On the contrary, according to the analytic method, to solve a problem, one looks for some hypothesis that is a sufficient condition for solving the problem, that is, such that a solution to the problem can be deduced from the hypothesis. The hypothesis is obtained from the problem, and possibly other data already available, by some non-deductive rule (induction, analogy, metaphor, etc.). The hypothesis must be plausible, that is, such that the arguments for it are stronger than the arguments against it, on the basis of experience. But the hypothesis, in turn, is a problem that must be solved, and it is solved in the same way. That is, one looks for some hypothesis that is a sufficient condition for solving the problem posed by the previous hypothesis, it is obtained from the latter, and possibly other data already available, by some non-deductive rule, and must be plausible. And so on[29].

An example of use of the analytic method is Newton's discovery of the propositions of his *Principia Mathematica*.

In fact, Newton says that these propositions «were invented by analysis»[30]. That is, by the analytic method. The latter is the method of «the mathematicians of the last age» who «have very much improved analysis» and «stop there», because they «think they have solved a problem when they have only resolved it», and «by this means the method of synthesis», that is, the axiomatic method, is «almost laid aside»[31].

[27] N. Bourbaki, *Elements of Mathematics: Theory of Sets*, Springer, Cham 2004, p. V.
[28] *Ibidem*.
[29] For more on the analytic method, see C. Cellucci, *The Making of Mathematics: Heuristic Philosophy of Mathematics*, Springer, Cham 2022, chapter 5.
[30] I. Newton, *MS Add. 3698, f. 101*, in B. Cohen, *Introduction to Newton's 'Principia'*, Cambridge University Press, Cambridge 1971, pp. 292-294, p. 294.
[31] *Ibidem*.

A recent example of use of the analytic method is the solution to Fermat's Last Problem. First, Ribet gave a solution to the problem using the Taniyama-Shimura hypothesis that every elliptic curve over the rationals is modular. Then Wiles and Taylor gave a solution to the problem posed by the Taniyama-Shimura hypothesis using other hypotheses. The latter, in turn, require other hypotheses, and so on.

The analytic method involves a reversal in the direction of mathematical research with respect to the axiomatic method. According to it, mathematical research does not consist in deducing theorems from axioms, but in obtaining hypotheses from problems in order to solve them, and in proving that the hypotheses are plausible.

Contrary to a widespread opinion, even Euclid thought that the axiomatic method is not the method of mathematics, it is only a method of teaching, the method of mathematics is the analytic method.

Euclid used the axiomatic method in the *Elements*, which was a textbook intended «to provide the student with an introduction»[32].

But, as we will see below, in his research work, which was part of the Analytic Corpus, Euclid proceeded by the analytic method.

That the axiomatic method is not the method of mathematics, it is only a method of exposition, is also affirmed by several contemporary mathematicians.

Thus, Rota says that the axiomatic method is not a method of research but only a «method of presentation of mathematics»[33]. So, it is only «a style of exposition»[34]. We must guard against «confusing mathematics with the axiomatic method for its presentation», in particular, we must guard against thinking that «mathematicians use the axiomatic method in solving problems and proving theorems», and hence that «the axiomatic method is a basic instrument of discovery»[35].

Thom says that «during the past few years the importance of axiomatization as an instrument of systematization and discovery has been much emphasized. As a method of systematizing, it is certainly effective; as for discovery, the matter is more doubtful»[36]. In fact, «no new theorem of any importance came out of the immense effort

[32] Proclus, *In primum Euclidis Elementorum librum commentarii*, 71.22-23, ed. Friedlein.
[33] G.-C. Rota, *Indiscrete Thoughts*, Birkhäuser, Boston 1997, p. 112.
[34] Ivi, p. 142.
[35] Ivi, p. 96.
[36] R. Thom, *'Modern' Mathematics. An Educational and Philosophic Error?*, «American Scientist» 59/6 (1971), pp. 695-699, p. 697.

at systematization of Nicolas Bourbaki»[37]. Discovery calls for other kinds of processes, «such as analogy»[38].

Hersh says that «a naive non-mathematician» who «looks into Euclid» and «observes that axioms come first», understandably «concludes that in mathematics, axioms come first. First your assumptions, then your conclusions, no? But anyone who has done mathematics knows what comes first – a problem»[39]. In mathematics, «problems, and solutions come first», therefore «the view that mathematics is in essence derivations from axioms is backward. In fact, it's wrong»[40].

## 7. *Approaches and Methods*

I have said that the change from the top-down approach to the bottom-up approach involves a change in the method of mathematics from the axiomatic method to the analytic method. This is because the top-down approach is inherently connected to the axiomatic method, while the bottom-up approach is inherently connected to the analytic method.

Thus, in the top-down approach, Einstein's general theory of relativity applied the Riemann geometry, which Riemann had created by the axiomatic method, to the physics of space and time.

In the bottom-up approach, Newton created calculus starting from problems in physics, and discovered the propositions of *Principia Mathematica* by the analytic method.

By its very origin, mathematics created by the bottom-up approach is likely to be applicable to a non-mathematical field. On the contrary, there is no guarantee that mathematics created with no application in mind, in particular, created by the axiomatic method, is applicable to a non-mathematical field.

Bourbaki says that one might ask «why do such applications ever succeed» but, «fortunately for us, the mathematician does not feel called upon to answer such questions»[41].

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[37] Ivi, pp. 697-698.
[38] Ivi, p. 699.
[39] R. Hersh, *What Is Mathematics, Really?*, cit., p. 6.
[40] *Ibidem*.
[41] N. Bourbaki, *Foundations of Mathematics for the Working Mathematician*, «The Journal of Symbolic Logic» 14 (1949), pp. 1-8, p. 2.

## 8. *Mainstream Philosophy of Mathematics*

The change from the axiomatic method to the analytic method requires a change in the philosophy of mathematics, from mainstream philosophy of mathematics to heuristic philosophy of mathematics.

Mainstream philosophy of mathematics is the dominant philosophy of mathematics today, because it is the philosophy of mathematics of analytic philosophy, which is the dominant philosophy.

The origin of mainstream philosophy of mathematics is Frege, who is also «the undisputed father of "analytic philosophy"», the «mainstream tradition in twentieth-century philosophy»[42].

Mainstream philosophy of mathematics is based on the following assumptions.

(1) The philosophy of mathematics cannot deal with the making of mathematics, in particular discovery. For discovery is a subjective process, and «we are unable to unite the inner states experienced by different people in one consciousness and so compare them»[43].

(2) The philosophy of mathematics can only deal with finished mathematics, namely mathematics as presented in books, journals, lectures. For finished mathematics is objective, therefore for each judgment we can give «the justification for making the judgment»[44].

(3) The philosophy of mathematics cannot contribute to the advancement of mathematics. In fact, «there are no new truths in my work»[45].

(4) The method of mathematics is the axiomatic method, so mathematics is theorem proving by the axiomatic method. For in mathematics one «starts from propositions that are accepted as true», the axioms, and arrives «via chains of inferences to the theorem»[46].

(5) Mathematics is a body of truths. For mathematics is «a system of

[42] T. Burge, *Gottlob Frege. Some Forms of Influence*, in M. Beaney (ed.), *The Oxford Handbook of the History of Analytic Philosophy*, Oxford University Press, Oxford 2013, pp. 355-382, p. 356.
[43] G. Frege, *Posthumous Writings*, Blackwell, Oxford 1979, p. 4.
[44] G. Frege, *The Foundations of Arithmetic: A Logico-Mathematical Enquiry Into the Concept of Number*, Harper, New York 1960, p. 3.
[45] G. Frege, *Begriffsschrift, a Formula Language, Modeled Upon That of Arithmetic, for Pure Thought*, in J. van Heijenoort (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Harvard University Press, Harvard 1967, pp. 5-82, p. 6.
[46] G. Frege, *Posthumous Writings*, cit., p. 204.

truths that are connected with one another by» deductive «inference»[47].

(6) The logic of mathematics is deductive logic, in particular, mathematical logic. For «there is no such thing as a peculiarly» mathematical «mode of inference that cannot be reduced to the general» deductive «inference-modes of» mathematical «logic»[48].

## 9. *Mainstream Philosophy of Mathematics and Incompleteness Theorems*

Despite being the dominant philosophy of mathematics today, mainstream philosophy of mathematics has serious defects.

A serious defect is that mainstream philosophy of mathematics is incompatible with the incompleteness theorems, therefore its assumptions (4) – (6) are not valid. This can be seen as follows.

(4) According to mainstream philosophy of mathematics, the method of mathematics is the axiomatic method, so mathematics is theorem proving by the axiomatic method. But, by Gödel's first incompleteness theorem, for any consistent, sufficiently strong, formal system, there are sentences of the system that are true but cannot be deduced from the axioms of the system. Therefore, the method of mathematics cannot be the axiomatic method.

Against this, Curry argues that Gödel's first incompleteness theorem only implies that no single formal system «can exhaust mathematics»[49]. But even if «the concept of intuitively valid proof cannot be exhausted by any single formalization», it can be exhausted by an infinitely growing sequence of formalizations, and «mathematical proof is precisely that sort of growing thing which the intuitionists have postulated for certain infinite sets»[50].

But this argument is not valid. A proof could be a growing thing only if there were a formalism that embraced all the steps in the infinitely growing sequence of formalizations. But, as Gödel argues, «there cannot exist any formalism which would embrace all these steps»[51].

(5) According to mainstream philosophy of mathematics, math-

[47] Ivi, p. 205.
[48] G. Frege, *Collected Papers on Mathematics, Logic, and Philosophy*, Blackwell, Oxford 1984, p. 113.
[49] H. Curry, *Foundations of Mathematical Logic*, Dover, Mineola 1977, p. 14.
[50] Ivi, p. 15.
[51] K. Gödel, *Collected Works*, Oxford University Press, Oxford 1986-2002, vol. II, p. 151.

ematics is a body of truths. But, by Gödel's second incompleteness theorem, for any consistent, sufficiently strong, formal system, it is impossible to demonstrate, by absolutely reliable means, that the axioms of the system are consistent, a fortiori that they are true, and hence that the theorems are true. Therefore, mathematics cannot be said to be a body of truths.

Against this, it could be argued that, if mathematics cannot be said to be a body of truths, then Gödel's result, being a mathematical theorem, cannot be said to be true. Therefore, the conclusion that, by Gödel's result, mathematics cannot be said to be a body of truths, is unjustified.

But this argument is not valid. For the conclusion that, by Gödel's result, mathematics cannot be said to be a body of truths, does not depend on the assumption that Gödel's result can be said to be true. It is a reductio ad absurdum. Indeed, suppose that mathematics can be said to be a body of truths. Then Gödel's result, being a mathematical theorem, can be said to be true. But, by Gödel's result, mathematics cannot be said to be a body of truths. Contradiction. Therefore, by reductio ad absurdum, we conclude that mathematics cannot be said to be a body of truths.

(6) According to mainstream philosophy of mathematics, the logic of mathematics is deductive logic, in particular, mathematical logic. But, by the strong incompleteness theorem for second order logic, there are no rules capable of deducing all second-order logical consequences of a given set of premises[52]. Therefore, the logic of mathematics cannot be deductive logic.

Prawitz says: «Mathematical knowledge is obtained by deductive inferences from truths that are considered to be obvious»[53].

Prawitz's view of mathematical knowledge is like the view that Lakatos calls «the Euclidean programme», according to which all mathematical «knowledge can be deduced from a finite set of trivially true propositions»[54].

But this view is not valid. For the initial premises from which math-

[52] See, for example, C. Cellucci, *The Theory of Gödel*, Springer, Cham 2022, section 9.6.

[53] D. Prawitz, *The Status of Mathematical Knowledge*, in E. Ippoliti-C. Cozzo (eds.), *From a Heuristic Point of View*, Cambridge Scholars Publishing, Newcastle upon Tyne 2014, pp. 73-90, p. 84.

[54] I. Lakatos, *Philosophical Papers*, Cambridge University Press, Cambridge 1978, vol. II, p. 4.

ematical knowledge is supposed to be deduced are not obvious. In fact, the initial premises of calculus and set theory even led to paradoxes.

Moreover, deductive inferences are non-ampliative, that is, the conclusion is contained in the premises. Therefore, this view implies that all mathematical knowledge consists of obvious truths. But this is patently implausible.

## 10. *Mainstream Philosophy of Mathematics and Mathematics*

Another serious defect is that mainstream philosophy of mathematics is inadequate to mathematics because its assumptions (1) – (3) are not valid. This can be seen as follows.

(1) According to mainstream philosophy of mathematics, the philosophy of mathematics cannot deal with the making of mathematics, in particular discovery. This implies that discovery cannot be based on logic but only on illumination.

This is also the opinion of several mathematicians.

Thus, Dieudonné says that it is impossible to deal with how mathematicians «arrived at their results», because «what goes on in a creative mind never has a rational "explanation", in mathematics any more than elsewhere. All that we know is that it» entails «sudden "illuminations", and a "formalizing" of what these have revealed»[55].

Byers says that mathematical discovery takes place when «a light has suddenly illuminated something that was formerly obscure» and the mathematician «has a "Eureka" moment», so mathematical discovery «is connected to, even based upon, illumination»[56]. Illumination «is not a logical process»[57]. For «logical arguments do not generate ideas», in fact «logic organizes, stabilizes and communicates ideas but the idea exists prior to the logical formulation»[58]. Therefore, «mathematics transcends logic»[59].

[55] J. Dieudonné, *Mathematics – The Music of Reason*, Springer, Cham 2013, p. 27.
[56] W. Byers, *The Blind Spot: Science and the Crisis of Uncertainty*, Princeton University Press, Princeton 2011, p. 41.
[57] W. Byers, *Can You Say What Mathematics Is?*, in B. Sriraman (ed.), *Humanizing Mathematics and Its Philosophy: Essays Celebrating the 90th Birthday of Reuben Hersh*, Springer, Cham 2017, pp. 45-60, p. 54.
[58] W. Byers, *How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics*, Princeton University Press, Princeton 2007, p. 259.
[59] Ivi, p. 26.

Wiles says that the making of mathematics is like a journey through a dark unexplored mansion, «one enters the first room of the mansion and it's dark. Completely dark. One stumbles around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated»[60]. These breakthroughs «are the culmination of, and couldn't exist without, the many months of stumbling around in the dark that precede them»[61].

The view that mathematical discovery cannot be based on logic but only on illumination relegates discovery to the sphere of irrationality, because no rational account can be given of illumination.

As Russell says, the belief in illumination is the «belief in the possibility of a way of knowledge which may be called revelation or insight or intuition, as contrasted with sense, reason, and analysis, which are regarded as blind guides leading to the morass of illusion»[62].

But this belief is unfounded, in fact the opposite is true. Rather than sense, reason, and analysis, it is intuition that is a blind guide leading to the morass of illusion. For intuition is unreliable and inadequate as a basis for mathematics.

The view that mathematical discovery cannot be based on logic but only on illumination conflicts with several historical cases.

In particular, Greek mathematicians had already invented a method of discovery, the analytic method, and they used it as a basis for their making of mathematics, and even reported their processes of discovery by publishing their analyses.

As Pappus tells us, their analyses were published in «the so-called Analytic Corpus», a body of works of the golden age of Greek geometry that is «a special resource prepared for the use of those who, after going through the ordinary elements, want to acquire a power in geometry of solving problems set to them»[63]. The Analytic Corpus «is the work of three men, Euclid, the author of the *Elementa*, Apollonius of Perga, and Aristaeus the Elder, and proceeds by the method of analysis»[64]. That is, by the analytic method.

[60] S. Singh, *Fermat's Enigma: The Epic Quest to Solve the World's Greatest Mathematical Problem*, Anchor Books, New York 1997, pp. 236-237.
[61] Ivi, p. 237.
[62] B. Russell, *Mysticism and Logic*, Routledge, London 1994, p. 27.
[63] Pappus, *Collectio*, VII, 634.3–7, ed. Hultsch.
[64] Ivi, VII, 634.8–11.

Against this, it could be argued that, by publishing analyses, textbooks and articles would become much longer. But, as Lakatos says, «the answer to this pedestrian argument is: let us try»[65].

The philosophy of mathematics must deal with the making of mathematics, in particular discovery, because only so it can provide an explanation of the mathematical process.

Dealing with discovery is also important for the making of mathematics. For it can lead to a development of method.

Even Frege admits that «a development of method, too, furthers science. Bacon, after all, thought it better to invent a means by which everything could easily be discovered than to discover particular truths, and all great steps of scientific progress in recent times have had their origin in an improvement of method»[66].

Dealing with discovery is also important for the teaching of mathematics. For it can be essential for the understanding of mathematics.

As Kline says, «the insistence on a deductive approach deceives the student» because «he is led to believe that mathematics is created by geniuses who start with axioms and reason directly from the axioms to the theorems»[67]. The «concentration on the deductive approach omits the real activity»[68]. Therefore, it «is often so artificial that is meaningless»[69].

(2) According to mainstream philosophy of mathematics, the philosophy of mathematics can deal with finished mathematics. But this contrasts with the fact that mainstream philosophy of mathematics cannot deal with several basic aspects of mathematics, such as mathematical objects, demonstrations, definitions, diagrams, and notations[70].

For example, mainstream philosophy of mathematics cannot deal with diagrams. For according to it, the method of mathematics is the axiomatic method, so everything in mathematics must be based on deduction from axioms, therefore diagrams have no role in mathematics.

[65] I. Lakatos, *Proofs and Refutations: The logic of Mathematical Discovery*, Cambridge University Press, Cambridge 1976, p. 144.

[66] G. Frege, *Begriffsschrift*, cit., p. 6.

[67] M. Kline, *Logic Versus Pedagogy*, «The American Mathematical Monthly» 77 (1970), pp. 264-282, pp. 273-274.

[68] Ivi, p. 272.

[69] *Ibidem*.

[70] See C. Cellucci, *The Making of Mathematics*, cit.

Thus, Hilbert says that figures «can easily be misleading»[71]. Therefore, «we will never rely on them»[72]. A «theorem is only proved when the proof is completely independent of the figure»[73].

It is emblematic that, at a 1959 conference on the need for reform in French education, the leading Bourbaki mathematician Dieudonné rose to his feet and shouted: Down with Euclid! Death to triangles! This became the slogan of Bourbaki.

Needham even says that it would be unfair and irrational if there were a law that prescribed, «Music must never be listened to or performed», but «in our society of mathematicians we have such a law. It is not a written law», but «it says, "Mathematics must not be visualized"», and in fact «over the last hundred years the honour of visual reasoning in mathematics has been besmirched»[74].

Since mainstream philosophy of mathematics cannot deal with several basic aspects of mathematics, it presents a distorted image of mathematics.

(3) According to mainstream philosophy of mathematics, the philosophy of mathematics cannot contribute to the advancement of mathematics.

Indeed, according to it, the task of the philosophy of mathematics is different from that of mathematics. While mathematics advances knowledge, the philosophy of mathematics should deal only with questions that make no difference in practice.

Thus, Wagner says that the philosophy of mathematics «should have no doctrinal or practical impact on mathematics at all»[75]. Mathematics «gives rise to substantive perplexities» but «the philosophical solution does not change our practices»[76].

So, however, the philosophy of mathematics becomes a marginal and ultimately irrelevant subject. If the philosophy of mathematics is to be relevant, it must contribute to the advancement of mathematics.

In fact, in the past, the philosophy of mathematics has contrib-

[71] D. Hilbert, *Die Grundlagen der Geometrie*, in M. Hallett-U. Majer (eds.), *David Hilbert's Lectures on the Foundations of Geometry 1891-1902*, Springer, Cham 2004, pp. 72-123, p. 75.

[72] Ivi, pp. 540-602, p. 541.

[73] Ivi, p. 75.

[74] T. Needham, *Visual Complex Analysis*, Oxford University Press, Oxford 1997, p. VII.

[75] S. Wagner, *Arithmetical Fiction*, «Pacific Philosophical Quarterly» 63 (1982), pp. 255-269, p. 267.

[76] *Ibidem*.

uted to it. For example, Berkeley contributed to the advancement of calculus by pointing out some critical questions that had to be addressed to obtain an adequate formulation.

Even a strong supporter of mainstream philosophy of mathematics like Robinson admits that «the vigorous attack directed by Berkeley against the foundations of the calculus in the forms then proposed is, in the first place, a brilliant exposure of their logical inconsistencies»[77].

## 11. *Requirements for an Adequate Philosophy of Mathematics*

From the defects of mainstream philosophy of mathematics described above, it follows that an adequate philosophy of mathematics must be compatible with the incompleteness theorems and must satisfy the following requirements.

(1) It must deal with the making of mathematics, in particular discovery. For only so it can provide an explanation of the mathematical process.

(2) It must deal with finished mathematics, including all basic aspects of mathematics. For only so it can avoid presenting a distorted image of mathematics.

(3) It must contribute to the advancement of mathematics. For only so it can avoid becoming a marginal and ultimately irrelevant subject.

## 12. *The Philosophy of Mathematical Practice*

It might be thought that the requirements for an adequate philosophy of mathematics are satisfied by the philosophy of mathematical practice.

But it is not so. For the philosophy of mathematical practice is not opposed to but continuous with mainstream philosophy of mathematics.

Thus, Carter says that «typically» the philosophy of mathematical practice «is characterized» in a way «not opposed to, but rather as an extension of traditional, or mainstream philosophy of mathe-

[77] A. Robinson, *Non-Standard Analysis*, North-Holland, Amsterdam 1966, p. 280.

matics»[78]. In fact, for the philosophy of mathematical practice «it is important to maintain relations with the mainstream philosophers of mathematics»[79].

In particular, like mainstream philosophy of mathematics, the philosophy of mathematical practice assumes that the logic of mathematics is deductive logic, notably mathematical logic.

Thus, Mancosu says that the philosophy of mathematical practice rejects the polemic of heuristic philosophy of mathematics «against the ambitions of mathematical logic as a canon for philosophy of mathematics»[80]. According to heuristic philosophy of mathematics, «mathematical logic cannot provide the tools for an adequate analysis of mathematics and its development»[81]. On the contrary, the philosophy of mathematical practice does not consider mathematical logic to be «ineffective», even «in dealing with the questions concerning the dynamics of mathematical discovery»[82].

This is because, like mainstream philosophy of mathematics, the philosophy of mathematical practice is part of analytic philosophy, which assumes that philosophy is not an inquiry aimed at acquiring knowledge, but only at understanding what we already know.

In fact, Mancosu says that, like mainstream philosophy of mathematics, the philosophy of mathematical practice does «not not dismiss the analytic tradition in philosophy of mathematics»[83].

A break with the analytic tradition in philosophy of mathematics is made only by heuristic philosophy of mathematics.

Since the philosophy of mathematical practice is continuous with mainstream philosophy of mathematics, it shares its defects.

## 13. *Heuristic Philosophy of Mathematics*

The requirements for an adequate philosophy of mathematics are only satisfied by heuristic philosophy of mathematics.

[78] J. Carter, *Introducing the Philosophy of Mathematical Practice*, Cambridge University Press, Cambridge 2024, p. 11.

[79] Ivi, p. 63.

[80] P. Mancosu, *Introduction*, in P. Mancosu (ed.), *The Philosophy of Mathematical Practice*, Oxford University Press, Oxford 2008, pp. 1-21, p. 4.

[81] Ivi, p. 5.

[82] Ivi, p. 4.

[83] Ivi, p. 18.

The origin of heuristic philosophy of mathematics is Lakatos's unpublished PhD thesis[84]. However, not all the assumptions on which heuristic philosophy of mathematics is based can be found in Lakatos[85].

Heuristic philosophy of mathematics is based on the following assumptions which, for comparison, are stated parallel to the assumptions of mainstream philosophy of mathematics.

(1) The philosophy of mathematics can deal with the making of mathematics, in particular discovery.

(2) The philosophy of mathematics can also deal with finished mathematics.

(3) The philosophy of mathematics can contribute to the advancement of mathematics.

(4) The method of mathematics is the analytic method, so mathematics is problem solving by the analytic method.

(5) Mathematics is a body of problems and solutions that are plausible.

(6) The logic of mathematics is the analytic method.

## 14. *Heuristic Philosophy of Mathematics and Incompleteness*

Heuristic philosophy of mathematics does not have the defects of mainstream philosophy of mathematics. Indeed, the assumptions (4) – (6) of heuristic philosophy of mathematics are compatible and even confirmed by the incompleteness theorems. This can be seen as follows.

(4) According to heuristic philosophy of mathematics, the method of mathematics is the analytic method, so mathematics is problem solving by the analytic method. This is confirmed by Gödel's first incompleteness theorem. For in the analytic method the solution to a problem is obtained by hypotheses not necessarily belonging to the same part of mathematics as the problem, and Gödel's first incompleteness theorem implies that solving a problem of a given part of mathematics may require hypotheses from other parts.

(5) According to heuristic philosophy of mathematics, mathematics is a body of problems and solutions that are plausible. This

[84] See I. Lakatos, *Essays in the Logic of Mathematical Discovery*, PhD Thesis, University of Cambridge, Cambridge 1961.

[85] On Lakatos' original formulation and its limitations, see C. Cellucci, *The Making of Mathematics*, cit., sections 3.2-3.3.

is confirmed by Gödel's second incompleteness theorem. For in the analytic method, the hypotheses for the solution to a problem are plausible but cannot be said to be true, and Gödel's second incompleteness theorem implies that in general no solution to a problem can be said to be true.

(6) According to heuristic philosophy of mathematics, the logic of mathematics is the analytic method. This is confirmed by the strong incompleteness theorem for second-order logic. For in the analytic method, deductive rules are not a closed set given once and for all, but an open set that can always be extended, and the strong incompleteness theorem for second-order logic implies that deductive rules cannot be a closed set given once and for all.

## 15. *Heuristic Philosophy of Mathematics and Mathematics*

Moreover, heuristic philosophy of mathematics is adequate to mathematics because its assumption (1) – (3) are valid. This can be seen as follows.

(1) According to heuristic philosophy of mathematics, the philosophy of mathematics can deal with the making of mathematics, in particular discovery. This assumption is valid because heuristic philosophy of mathematics bases the making of mathematics on the analytic method, which since antiquity has been the fundamental method of discovery.

(2) According to heuristic philosophy of mathematics, the philosophy of mathematics can also deal with finished mathematics. This assumption is valid because heuristic philosophy of mathematics can deal with all basic aspects of mathematics, such as mathematical objects, demonstrations, definitions, diagrams, and notations[86]. For example, it can deal with diagrams because, according to it, diagrams are figures such that, on the basis of data acquired from them, it is possible to formulate hypotheses that are sufficient conditions for solving problems by the analytic method.

(3) According to heuristic philosophy of mathematics, the philosophy of mathematics can contribute to the advancement of mathematics. This assumption is valid because heuristic philosophy of mathematics can contribute to discover new results since it bases the

[86] See C. Cellucci, *The Making of Mathematics*, cit., chapters 9-13.

making of mathematics on the analytic method which, as already said, since antiquity has been the fundamental method of discovery.

## 16. *Conclusion*

From what has been said above it is possible to conclude that the developments that are coming in physics and biology really involve three changes in our understanding of mathematics.

(1) The developments involve a change in the approach of mathematics to physics and biology, from the top-down approach to the bottom-up approach. For the developments have a high degree of complexity, and only mathematics created by the bottom-up approach can deal with such a degree of complexity.

(2) The developments involve a change in the method of mathematics, from the axiomatic method to the analytic method. For while the top-down approach is inherently connected to the axiomatic method, the bottom-up approach is inherently connected to the analytic method.

(3) The developments involve a change in the philosophy of mathematics, from mainstream philosophy of mathematics to heuristic philosophy of mathematics. For only heuristic philosophy of mathematics can address the complexity and dynamism that the bottom-up approach entails.

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